HEAT AND MASS TRANSFER IN A HYDMAGNETIC FLOW BETWEEN TWO PARALLEL PERMEABLE PLATES

Nabil T. M. Eldabe, Mona A. A. Mohamed
Department of Mathematics, Faculty of Education
Ain Shams University
Heliopolis, Cairo, Egypt

Abstract

An analytical study of magnetohydrodynamic free-convective and mass-transfer flow between two parallel permeable plates, is presented, tacking into account the Eckert number effects. The solution of the problem is obtained using perturbation method with Eckert number $E(<< 1)$ as perturbation parameter. Analytical expressions are given for the velocity distribution, temperature field, skin-friction, heat transfer and mass transfer. The detailed study has been made to know the effects of the various parameters entering into the problem.

Introduction

Effects of magnetic field on doubly diffusive fluid systems find application in various branches of modern science like biochemistry, oceanography, stellar convection, etc. Eldabe [1] have made an interesting analysis of the magnetohydrodynamic unsteady free convective flow through a porous medium bounded by an infinite vertical porous plate. Also, magnetohydrodynamic flow and heat transfer in a viscoelastic incompressible fluid confined between a horizontal stretching sheet and a parallel porous wall was presented by Eldabe and Sallam [2]. In a similar manner, Singh [3], Graham and Hunt [4] and Srivastava and Sharma [5] have studied some problems in this field.

However, all of the above research dealing with the Newtonian and non-Newtonian fluids concentrates only on the heat transfer problems. None of them deals with the much more complicated problem which involves both the heat and the mass transfer in the electrically conducting fluid. As a result, this research attempts to solve this much more complicated problem, taking into account the thermal-diffusion effect. For the governing equations of the problem, closed form analytic solutions are obtained and evaluated numerically for different parameters, such as the Prandtl number $P$, the magnetic parameter $M$, the modified Schmidt number $S_c$, Soret number, and the Eckert number. The contributions of these parameters to the heat and mass transfer characteristics are shown to be quite significant.
Analysis

Consider a steady two dimensional hydromagnetic flow of a viscous, incompressible and electrically-conducting fluid confined between two parallel plates. The lower plate fixed, and the upper moves with a uniform velocity in its own plane. The temperature and salinities at the lower y=0 are $T_0$ and $C_0$ and at the upper y=h are $T_1$ and $C_1$, respectively, y-axis being taken as vertical. A uniform vertical magnetic field of strength $B_0$ is applied on the system. The induced magnetic field is assumed to be negligible. The equations governing the motion of a fluid flowing between two parallel plates, in the usual notation, are:

$$v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\lambda(T - \bar{T}) + g\lambda^*(C - \bar{C}) - \frac{\sigma B_0^2}{\rho} u \quad (1)$$

$$v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p} \left( \frac{\partial u}{\partial y} \right)^2 \quad (2)$$

$$v \frac{\partial C}{\partial y} = \beta \frac{\partial^2 T}{\partial y^2} + \gamma \frac{\partial^2 C}{\partial y^2} \quad (3)$$

$$\frac{\partial v}{\partial y} = 0 \quad (4)$$

where $(u,v)$ are the velocity components of the fluid in x and y directions respectively, $\rho$ is the density of the fluid, $\nu = \frac{\mu}{\rho}$, is the kinematic viscosity, $\mu$ is the coefficient of viscosity, $T$ is the temperature, $\bar{C}$ is the solute concentration, $g$ is the acceleration due to gravity, $\bar{T}$ is the mean of $T_0$ and $T_1$, $\bar{C}$ is the mean of $C_0$ and $C_1$, $\lambda$ is the coefficient of volume expansion, $\lambda^*$ is the coefficient of solute mass expansion, $\alpha, \beta, \gamma$ are the thermal diffusivity of the fluid medium, $\gamma$ is the molecular diffusivity, $\sigma$ is the electrical conductivity of the fluid and $C_p$ is the specific heat.

Equations (1)-(4) express the conservation of momentum, heat, solute and mass. Also, in equation (2), due to negligible induced magnetic field, the heat due to Joule dissipation is also assumed to be negligible. Equation (4) shows that $v$ is either a constant or a function of time and as we assume a steady state the suction velocity $v$ will takes the constant value $v_0$. Also the two plates are infinite in extent, hence all quantities are functions of y only. Equations (1)-(3) reduce to

$$v_0 \frac{du}{dy} = \nu \frac{d^2 u}{dy^2} + g\lambda(T - \bar{T}) - g\lambda^*(C - \bar{C}) - \frac{\sigma B_0^2}{\rho} u \quad (5)$$

$$v_0 \frac{dT}{dy} = \alpha \frac{d^2 T}{dy^2} + \frac{\nu}{C_p} \left( \frac{du}{dy} \right)^2 \quad (6)$$

$$v_0 \frac{dC}{dy} = \beta \frac{d^2 T}{dy^2} + \gamma \frac{d^2 C}{dy^2} \quad (7)$$

The appropriate boundary conditions for the problem are:
\[ u = 0, \ T = T_0, \ C = C_0, \ \text{at} \ y = 0 \]
\[ u = u_0, \ T = T_1, \ C = C_1, \ \text{at} \ y = h \] (8)

where \( u_0 \) is the uniform velocity of the upper plate in its own plane.

Introducing the following non-dimensional quantities
\[ u' = \frac{u}{u_0}, \ y' = \frac{y}{u_0}, \ \theta = \frac{T - T_0}{T_0 - T_1}, \ C' = \frac{C - C_0}{C_0 - C_1}, \ n = \frac{T_1 - T}{T_0 - T}, \ m = \frac{C_1 - C}{C_0 - C}, \ S = \frac{h u_0}{\nu}, \ G = \frac{\nu \lambda (T_0 - T)}{u_0^2} \]
the Grashof number, \( G_c = \frac{\nu \lambda (C_0 - C)}{u_0^2} \) the modified Grashof number, \( P = \frac{\nu}{\alpha} \)
the Prandtl number, \( S_c = \frac{\nu}{\gamma} \) the Schmidt number, \( S_0 = \frac{\nu (T_0 - T)}{\nu (C_0 - C)} \) the Soret number,
\[ E = \frac{u_0^2}{C_p (T_0 - T)} \] the Eckert number, \( M = \frac{\nu B_0^2}{\rho u_0^2} \) the magnetic field parameter.

Equations [5]-[8] after substituting from equation (9) may be written in dimensionless form after dropping the dash mark.

\[ \frac{d^2 u}{dy^2} - \frac{du}{dy} = Mu - G\theta - G_cC, \] (10)

\[ \frac{d^2 \theta}{dy^2} - P \frac{d\theta}{dy} = -PE \left( \frac{du}{dy} \right)^2, \] (11)

\[ \frac{d^2 C}{dy^2} - S_c \frac{dC}{dy} = -S_0 S_c \frac{d^2 \theta}{dy^2}, \] (12)

with the boundary conditions:
\[ u = 0, \ \theta = 1, \ C = 1, \ \text{at} \ y = 0 \]
\[ u = 1, \ \theta = n, \ C = m, \ \text{at} \ y = s \] (13)

These equations are coupled and non-linear and hence very difficult to solve. To solve them, \( u, \ \theta \) and \( C \) are expanded in powers of \( E \), the Eckert number, the Eckert number for all incompressible fluids is always \( < < 1 \) [6]. Hence we take

\[ u(y) = u_0(y) + Eu_1(y) + O(E^2) \]
\[ \theta(y) = \theta_0(y) + E\theta_1(y) + O(E^2) \]
\[ C(y) = C_0(y) + E\theta_1(y) + O(E^2) \] (14)

Substituting (14) in (10)-(13), equating the coefficients of different powers of \( E \), we have the set of the following coupled linear equations:

\[ (D^2 - D - M)u_0 = -G\theta_0 - G_cC_0 \] (15)

\[ (D^2 - D - M)u_1 = -G\theta_1 - G_cC_0 \] (16)

\[ (D^2 - PD)\theta_0 = 0 \] (17)
\[(D^2 - PD)\theta_1 = -P(Du_0)^2\]  \hspace{1cm} (18)

\[(D^2 - S_cD)C_0 = -S_0S_cD^2\theta_0\]  \hspace{1cm} (19)

\[(D^2 - S_cD)C_1 = -S_0S_cD^2\theta_1\]  \hspace{1cm} (20)

with the corresponding boundary conditions

\[u_0 = 0, \ \theta_0 = 1, \ C_0 = 1, \ \text{at} \ y = 0\]  \hspace{1cm} (21)

\[u_0 = 1, \ \theta_0 = n, \ C_0 = m, \ \text{at} \ y = s\]  \hspace{1cm} (22)

\[u_1 = 0, \ \theta_1 = 0, \ C_1 = 0, \ \text{at} \ y = 0\]  \hspace{1cm} (22)

\[u_1 = 0, \ \theta_1 = 0, \ C_1 = 0, \ \text{at} \ y = s\]  \hspace{1cm} (22)

where \(D\) means \(\frac{d}{dy}\).

The solutions of (15)-(20) in virtue of the boundary conditions (21) and (22) are respectively given by:

\[\theta_0 = b_3e^{Py} + b_4\]  \hspace{1cm} (23)

\[C_0 = \left( b_8 + \frac{S_0S_c b_0b_3 P}{P - S_c} \right) e^{Scy} - \frac{S_0S_c b_3 P}{P - S_c} e^{Py} + b_9 - \frac{S_0S_c b_3 P(b_0 - b_5)}{b_5(P - S_c)}\]  \hspace{1cm} (24)

\[u_0 = b_{19}e^{yb_{16}} + b_{20}e^{yb_{17}} - b_{10}e^{Py} - b_{11}e^{Scy} + b_{12},\]  \hspace{1cm} (25)

\[\theta_1 = b_{39}e^{Py} + b_{50} - P[b_{38}e^{2yb_{16}} + b_{40}e^{2yb_{17}} + b_{41}e^{2Py} + b_{43}e^{2Scy} + b_{44}e^{2yb_{26}} - b_{45}e^{2yb_{28}} + b_{46}e^{yb_{30}} - b_{47}e^{yb_{32}} - b_{48}e^{yb_{34}} - b_{49}e^{yb_{35}}] \]  \hspace{1cm} (26)

\[C_1 = b_{79}e^{ySc} + b_{80} - b_{62}e^{Py} + b_{63}e^{2yb_{16}} + b_{64}e^{2yb_{17}} + b_{65}e^{2Py} + b_{66}e^{2ySc} + b_{67}e^{yb_{26}} - b_{68}e^{yb_{28}} + b_{69}e^{yb_{30}} - b_{70}e^{yb_{32}} - b_{71}e^{yb_{34}} - b_{72}e^{yb_{35}} \]  \hspace{1cm} (27)

\[u_1 = b_{106}e^{yb_{16}} - b_{107}e^{yb_{17}} + b_{84}e^{Py} - b_{85}e^{ySc} + b_{86}e^{2P} + b_{87}e^{2ySc} + b_{88}e^{2yb_{16}} + b_{89}e^{2yb_{17}} + b_{90}e^{yb_{26}} + b_{91}e^{yb_{28}} + b_{92}e^{yb_{30}} + b_{93}e^{yb_{32}} + b_{94}e^{yb_{34}} + b_{95}e^{yb_{36}} + b_{96} \]  \hspace{1cm} (28)

Hence, the expressions for \(u\), \(\theta\) and \(C\) can be derived from (23)-(28) and (14) as follows:

\[u = b_{108}e^{yb_{16}} + b_{109}e^{yb_{17}} + b_{110}e^{Py} - b_{111}e^{ySc} + E \{b_{86}e^{2Py} + b_{87}e^{2ySc} + b_{88}e^{2yb_{16}} + b_{89}e^{2yb_{17}} + b_{90}e^{yb_{26}} + b_{91}e^{yb_{28}} + b_{92}e^{yb_{30}} + b_{93}e^{yb_{32}} + b_{94}e^{yb_{34}} + b_{95}e^{yb_{36}} + b_{112}, \]  \hspace{1cm} (29)
\[ \theta = (b_3 + E b_{59}) e^{P y} - P E \left\{ b_{38} e^{2 v b_{16}} + b_{40} e^{2 v b_{17}} + b_{41} e^{2 P y} + b_{43} e^{2 y S_c} + b_{44} e^{v b_{26}} - b_{45} e^{v b_{28}} + b_{46} e^{v b_{30}} - b_{47} e^{v b_{32}} - b_{48} e^{v b_{34}} - b_{49} e^{v b_{36}} + (b_4 + E b_{60}) \right\} \]

\[ C = b_{81} e^{5 v S_c} - b_{82} e^{P y} + E \left\{ b_{63} e^{2 v b_{16}} + b_{64} e^{2 v b_{17}} + b_{65} e^{2 P y} + b_{66} e^{2 y S_c} + b_{67} e^{v b_{26}} - b_{68} e^{v b_{28}} + b_{69} e^{v b_{30}} - b_{70} e^{v b_{32}} - b_{71} e^{v b_{34}} - b_{72} e^{v b_{36}} + b_{83} \right\} \]

where \( b_0 - b_{112} \) are defined in the Appendix.

From the knowledge of the velocity, we can calculate the skin friction which is given in dimensionless form by

\[ \tau_{xy} = \frac{du}{dy}, \]

At the lower plate

\[ \tau_{xy} \big|_{y=0} = b_{117}, \]

and at the upper plate

\[ \tau_{xy} \big|_{y=5} = b_{122}, \]

Knowing the temperature field, it is important to know the Nusselt number and is given in non-dimensional form by

\[ N_u = \frac{d \theta}{dy}, \]

At the lower plate

\[ N_u \big|_{y=0} = b_{126}, \]

and at the upper plate

\[ N_u \big|_{y=5} = b_{132}, \]

Knowing the concentration field, it is important to know the rate of concentration and is given in dimensionless form by

\[ M_c = \frac{dC}{dy}, \]

At the lower plate

\[ M_c \big|_{y=0} = b_{137}, \]

and at the upper plate

\[ M_c \big|_{y=5} = b_{142}, \]

where, \( b_{117}, b_{122}, b_{126}, b_{132}, b_{137} \) and \( b_{142} \) are defined in the Appendix.
Results and Discussion

Heat and mass transfer in a Hydromagnetic flow between two parallel permeable plates represents a very important class of fluid mechanics problems. That flow is governed by five parameters, namely, the Prandtl number $P$, the magnetic parameter $M$, the Schmidt number $Sc$, the Soret number $So$, the Eckert number $Ec$. An insight into the effect of these parameters on the flow can be obtained by the study of the velocity distribution, temperature field and concentration. In figs. 1-3, for $M=1$, $Sc=0.3$, $So=0.5$, $Ec=0.1$, the velocity, the temperature and the concentration are plotted for several values of the prandtl number $P$, it is found that all of them increase with the increase of $P$.

![Graph showing velocity distribution](image)

fig. 1: Velocity distribution plotted against position in the case $M=1$, $Sc=0.3$, $So=0.3$, $G=5$, $Gc=5$, $Ec=0.1$
fig. 2: Temperature distribution plotted against position in the case $M=1$, $Sc=0.3$, $S_0=0.3$, $G=5$, $G_c=5$, $Ec=0.1$

fig. 3: Salinity distribution plotted against position in the case $M=1$, $Sc=0.3$, $S_0=0.3$, $G=5$, $G_c=5$, $Ec=0.1$
figs. 4-6, for $P=0.5$, $Sc=0.3$, $So=0.5$, $Ec=0.1$, are describe the effect of the magnetic parameter $M$, one can sees that all of the velocity, temperature and the concentration are decrease with the increase of $M$.

**Fig. 4:** Velocity distribution plotted against position in the case $P=0.5$, $Sc=0.3$, $So=0.3$, $G=5$, $G_c=5$, $Ec=0.1$

**Fig. 5:** Temperature distribution plotted against position in the case $P=0.5$, $Sc=0.3$, $So=0.3$, $G=5$, $G_c=5$, $Ec=0.1$
fig. 6: Salinity distribution plotted against position in the case $P=0.5$, $Sc=0.3$, $S0=0.3$, $G=5$, $Gc=5$, $Ec=0.1$

From figs.7-9, for $P=0.5$, $M=1$, $So=0.5$, $Ec=0.1$, it is found that both the velocity and the temperature are decrease with the increase of the schmidt number $Sc$ but the concentration decreases in the range from $y=0$ to $y=1.4$ while it increases in the range from $y=1.4$ to $y=2$

fig. 7: Velocity distribution plotted against position in the case $P=0.5$, $M=1$, $S0=0.3$, $G=5$, $Gc=5$, $Ec=0.1$
fig. 8: Temperature distribution plotted against position in the case $P=0.5$, $M=1$, $S_0=0.3$, $G=5$, $G_c=5$, $E_c=0.1$

fig. 9: Salinity distribution plotted against position in the case $P=0.5$, $M=1$, $S_0=0.3$, $G=5$, $G_c=5$, $E_c=0.1$

Figs. 10-12, for $P=0.5$, $M=1$, $Sc=0.3$, $Ec=0.1$, describe the effect of the Soret number $S_o$ on the velocity, the temperature and the concentration, it is found that all of them
increase with the increase of So

fig. 10: Velocity distribution plotted against position in the case P=0.5, M=1, Sc=0.3, G=5, Gc=5, Ec=0.1

fig. 11: Temperature distribution plotted against position in the case P=0.5, M=1, Sc=0.3, G=5, Gc=5, Ec=0.1
fig. 12: Salinity distribution plotted against position in the case $P=0.5$, $M=1$, $Sc=0.3$, $G=5$, $Ge=5$, $Ec=0.1$

Finally, from figs.13-15, for $P=0.5$, $M=1$, $So=0.5$, $Sc=0.3$, it is found that the effect of the Eckert number $Ec$ is to increase all of the velocity, the temperature and the concentration.

fig. 13: Velocity distribution plotted against position in the case $P=0.5$, $M=1$, $Sc=0.3$, $So=0.3$, $G=5$, $Ge=5$, $Ec=0.1$.
fig. 14: Temperature distribution plotted against position in the case $P=0.5$, $M=1$, $Sc=0.3$, $S\theta=0.3$, $G=5$, $Gc=5$.

fig. 15: Salinity distribution plotted against position in the case $P=0.5$, $M=1$, $Sc=0.3$, $S\theta=0.3$, $G=5$, $Gc=5$. 
Appendix

\[ b_0 = e^{PS} - 1, \quad b_3 = \frac{b_1}{b_0}, \]
\[ b_2 = b_0 - b_1, \]
\[ b_4 = \frac{b_2}{b_0}, \quad b_5 = e^{SSc} - 1, \]
\[ b_6 = m - 1, \quad b_7 = b_5 - b_6, \]
\[ b_8 = \frac{b_4}{b_5}, \quad b_9 = \frac{b_7}{b_5}, \]
\[ b_{10} = \frac{b_3}{P^2 - P - M} \left[ Gc - \frac{GcS_cScP}{P - S_c} \right], \]
\[ b_{11} = \frac{Gc}{S_c^2 - S_c - M} \left[ b_8 + \frac{S_cS_cS_cPb_2}{b_5(P - S_c)} \right], \]
\[ b_{12} = \frac{1}{M} \left[ Gb_4 + Gc \left\{ b_9 - \frac{b_4S_cS_cP}{b_5(P - S_c)}(b_0 - b_5) \right\} \right], \]
\[ b_{13} = b_{10} + b_{11} - b_{12}, \]
\[ b_{14} = b_{10}e^{PS} + b_{11}e^{S_c}b_2, \]
\[ b_{15} = \sqrt{1 + 4M}, \]
\[ b_{16} = \frac{1 - b_{14}}{2}, \]
\[ b_{17} = \frac{1 - b_{14}}{2}, \]
\[ b_{18} = e^{Sb_1}e^{Sb_16}, \]
\[ b_{19} = \frac{(b_{13}e^{Sb_17} - b_{14} - 1)/b_{18},}{b_{20} = b_{13} - b_{19},} \]
\[ b_{21} = (b_{16}b_{19})^2, \]
\[ b_{22} = (b_{17}b_{20})^2, \]
\[ b_{23} = (Pb_{10})^2, \]
\[ b_{24} = (S_c b_{11})^2, \]
\[ b_{25} = 2b_{16}b_{17}b_{19}b_{20}, \]
\[ b_{26} = b_{16} + b_{17}, \]
\[ b_{27} = 2Pb_{10}b_{17}b_{20}, \]
\[ b_{28} = P + b_{17}, \]
\[ b_{29} = 2PS_c b_{10}b_{11}, \]
\[ b_{30} = P + S_c, \]
\[ b_{31} = 2Pb_{10}b_{16}b_{19}, \]
\[ b_{32} = P + b_{16}, \]
\[ b_{33} = 2S_c b_{11}b_{16}b_{19}, \]
\[ b_{34} = S_c + b_{16}, \]
\[ b_{35} = 2S_c b_{11}b_{17}b_{20}, \]
\[ b_{36} = S_c + b_{17}, \]
\[ b_{37} = 2b_{16}(2b_{16} - P), \]
\[ b_{38} = b_{21}/b_{37}, \]
\[ b_{39} = 2b_{17}(2b_{17} - P), \]
\[ b_{40} = b_{22}/b_{39}, \]
\[ b_{41} = b_{23}/2P^2, \]
\[ b_{42} = 2S_c(2S_c - P), \]
\[ b_{43} = b_{24}/b_{42}, \]
\[ b_{44} = b_{25}/(b_{26} - P), \]
\[ b_{45} = b_{27}/[b_{28}(b_{28} - P)], \]
\[ b_{46} = b_{29}/(b_{30}P), \]
\[ b_{47} = b_{31}/[b_{32}(b_{32} - P)], \]
\[ b_{48} = (b_{33}b_{34} - P), \]
\[ b_{49} = b_{35}/[b_{36}(b_{36} - P)], \]
\[ b_{50} = b_{38} + b_{40} + b_{41} + b_{43} + b_{44}, \]
\[ b_{51} = b_{45} - b_{46} + b_{47} + b_{48} + b_{49}, \]
\[ b_{52} = P[b_{50}b_{51}], \]
\[ b_{53} = b_{38}e^{2Sb_{16}} + b_{40}e^{Sb_{17}}, \]
\[ b_{54} = b_{41}e^{2SP} + b_{43}e^{2SS_c}, \]
\[ b_{55} = b_{44}e^{Sb_{26}} - b_{45}e^{Sb_{28}}, \]
\[ b_{56} = b_{46}e^{Sb_{30}b_{47}e^{Sb_{32}}} \]
\[ b_{57} = b_{48}e^{Sb_{34}} + b_{49}e^{Sb_{36}}, \]
\[ b_{58} = (b_{53} + b_{54} + b_{55} + b_{56} - b_{57})P \]
\[ b_{59} = (b_{58} - b_{52})/b_{0}, \]
\[ b_{60} = (b_{52}P - b_{58})/b_{0b}, \]
\[ b_{61} = S_0S_c, \]
\[ b_{62} = \frac{Pb_{53}b_{51}}{Pc}, \]
\[ b_{63} = \frac{2b_{16}b_{19}b_{17}P}{2b_{16} - S_c}, \]
\[ b_{64} = \frac{2b_{16}b_{16}b_{14}b_{17}P}{b_{16} - S_c}, \]
\[ b_{65} = \frac{2Pb_{11}b_{19}P}{2P - S_c}, \]
\[ b_{66} = \frac{2Pb_{10}b_{11}P}{b_{16} - S_c}, \]
\[ b_{67} = \frac{b_{21}b_{41}b_{17}P}{b_{21} - S_c}, \]
\[ b_{68} = \frac{b_{26}b_{17}b_{19}P}{b_{26} - S_c}, \]
\[ b_{69} = \frac{b_{26}b_{14}b_{17}P}{b_{26} - S_c}, \]
\[ b_{70} = \frac{b_{27}b_{17}b_{19}P}{b_{27} - S_c}, \]
\[ b_{71} = \frac{b_{27}b_{17}b_{19}P}{b_{27} - S_c}, \]
\[ b_{72} = \frac{b_{28}b_{17}b_{19}P}{b_{28} - S_c}, \]
\[ b_{73} = b_{63} + b_{64} + b_{65} + b_{66} + b_{67}, \]
\[ b_{74} = b_{68} + b_{69} + b_{70} + b_{71} + b_{72}, \]
\[ b_{75} = b_{62} - b_{73} + b_{74}, \]
\[ b_{76} = b_{63}e^{2Sb_{16}} + b_{64}e^{2Sb_{17}} + b_{65}e^{2PS} + b_{66}e^{2SS_c} + b_{67}e^{Sb_{26}}, \]
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\[ b_{77} = b_{68}e^{2b_{8}e^{0.99}e^{2b_{30}} + b_{70}e^{3b_{22}} + b_{71}e^{3b_{34}} + b_{72}e^{3b_{58}}} \]
\[ b_{78} = b_{62}e^{PS} - b_{76} + b_{77} \]
\[ b_{80} = b_{78}e^{SS_{2} - b_{78}} \]
\[ b_{81} = b_{8} + \frac{S_{8}S_{9}b_{08}P}{b_{9}(P - S_{8})} + E_b_{79} \]
\[ b_{82} = \frac{S_{8}S_{9}b_{08}P}{P - S_{8}} + E_b_{82} \]
\[ b_{83} = b_{9} - \frac{S_{8}S_{9}P(b_{9} - b_{3})}{P - S_{8}} + E_b_{80} \]
\[ b_{84} = (G_{c}b_{0} - G_{c}b_{3})/(P^{2} - P - M) \]
\[ b_{86} = (G_{P}b_{31} - G_{c}b_{65})/(4P^{2} - 2P - M) \]
\[ b_{88} = (G_{P}b_{31} - G_{c}b_{63})/(4b_{16}^{2} - 2b_{16} - M) \]
\[ b_{90} = (G_{P}b_{34} - G_{c}b_{67})/(b_{26}^{2} - b_{26} - M) \]
\[ b_{92} = (G_{P}b_{46} - G_{c}b_{69})/(b_{30}^{2} - b_{30} - M) \]
\[ b_{94} = (G_{P}b_{71} - G_{c}b_{68})/(b_{34}^{2} - b_{34} - M) \]
\[ b_{96} = (G_{c}b_{60} + G_{c}b_{68})/M \]
\[ b_{98} = b_{88} + b_{80} + b_{90} + b_{91} \]
\[ b_{100} = b_{88} + b_{90} + b_{99} \]
\[ b_{101} = b_{84}e^{PS} - b_{85}e^{SS_{2} + b_{86}e^{2SP}} \]
\[ b_{102} = b_{87}e^{2SS_{2} + b_{88}e^{2b_{16} + b_{89}e^{2b_{17}}} \]
\[ b_{104} = b_{93}e^{3b_{32} + b_{94}e^{3b_{33} + b_{95}e^{3b_{35}} + b_{96}} \]
\[ b_{106} = (b_{105} - b_{100}e^{b_{17}})/b_{18} \]
\[ b_{108} = b_{19} + E_b_{86} \]
\[ b_{110} = -b_{10} + E_b_{84} \]
\[ b_{112} = b_{12} + E_b_{86} \]
\[ b_{114} = 2E[P_{b_{86}} + S_{c}b_{87} + b_{16}b_{88} + b_{17}b_{89}] \]
\[ b_{115} = E[b_{26}b_{90} + b_{28}b_{91} + b_{30}b_{92}] \]
\[ b_{116} = b_{32}b_{93} + b_{34}b_{94} + b_{35}b_{95} \]
\[ b_{117} = b_{13} + b_{14} + b_{15} + b_{16} \]
\[ b_{118} = b_{16}b_{108}e^{b_{15} + b_{17}b_{109}e^{b_{17} + P_{b_{110}}}e^{SP} - S_{c}b_{111}e^{SS_{2}}} \]
\[ b_{119} = 2E[P_{b_{86}}e^{2PS} + S_{c}b_{87}e^{2SS_{2} + b_{16}b_{88} + b_{17}b_{89}e^{2b_{17}}} \]
\[ b_{120} = E[b_{26}b_{90}e^{b_{28} + b_{28}b_{91}e^{b_{28} + b_{30}b_{92}e^{b_{30}}}] \]
\[ b_{121} = E[b_{32}b_{93}e^{b_{32} + b_{34}b_{94}e^{b_{34} + b_{35}b_{95}e^{b_{35}}}] \]
\[ b_{122} = b_{118} + b_{119} + b_{120} + b_{121} \]
\[ b_{123} = P/E(b_{59} - 2b_{16}b_{38} - 2b_{17}b_{40} - 2P_{b_{41}} \]
\[ b_{124} = P(b_{3} - 2E_{c}b_{43} - E_{b_{26}b_{44}} + E_{b_{28}b_{45}}) \]
\[ b_{125} = PE(b_{24} + b_{24}b_{48} + b_{36}b_{49} - b_{30}b_{46}) \]
\[ b_{126} = b_{123} + b_{124} + b_{125} \]
\[ b_{127} = P(b_{3} + E_{b_{59}}) \]
\[ b_{128} = 2b_{16}b_{38}e^{2b_{16} + b_{17}b_{40}e^{2b_{17} + P_{b_{41}}e^{2PS} + S_{c}b_{63}e^{2SS_{2}}} \]
\[ b_{129} = b_{26}b_{44}e^{b_{28} + b_{28}b_{45}e^{b_{30}b_{45}e^{2PS}} + b_{30}b_{45}e^{2b_{30}}} \]
\[ b_{130} = b_{32}b_{47}e^{b_{32} + b_{34}b_{48}e^{b_{34} + b_{36}b_{49}e^{b_{36}}}} \]
\[ b_{131} = b_{128} + b_{129} - b_{130} \]
\[ b_{132} = b_{127} - P/E_{b_{131}} \]
\[ b_{133} = S_{c}b_{81} - P_{b_{82}} + 2E_{b_{16}b_{63}} \]
\[ b_{134} = 2E_{b_{17}b_{64}} + 2EP_{b_{65}} + 2ES_{c}b_{66} \]
\[ b_{135} = E(b_{26}b_{67} - b_{28}b_{86} + b_{30}b_{69}) \]
\[ b_{136} = E(b_{32}b_{70} + b_{34}b_{71} + b_{36}b_{72}) \]
\[ b_{137} = b_{133} + b_{134} + b_{135} - b_{136} \]
\[ b_{138} = S_{c}b_{81}e^{SS_{2} - P_{b_{92}}e^{PS} + 2E_{b_{16}b_{63}}e^{2b_{86}}} \]
\[ b_{139} = 2E(b_{17}b_{84}e^{2b_{17} + P_{b_{65}}e^{2PS} + S_{c}b_{66}e^{2SS_{2}}} \]
\[ b_{140} = E(b_{26}b_{67}e^{b_{28} + b_{28}b_{68}e^{b_{28} + b_{30}b_{69}e^{b_{30}}}) \]
\[ b_{141} = E(b_{32}b_{70}e^{b_{32} + b_{34}b_{72}e^{b_{34} + b_{36}b_{72}e^{b_{36}}}}) \]
\[ b_{142} = b_{138} + b_{139} + b_{140} - b_{141} \]
References